ADITYA ENGINEERING COLLEGE (A)

VECTOR DIFFERENTIATION

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VECTORS & SCALARS

- Scalars are an abstraction of physical concepts like mass , which have only magnitude.

- Vectors are an abstraction of physical concepts like displacement and force, which have magnitude and direction

POSITION VECTORS

- A vector which is the displacement from the origin of coordinates to a point (x,y,z) can be written as

$$\vec{r} = xi + yj + zk$$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

KEY POINTS

- The displacement vector from r_1 point to r_2 point is r_1-r_2

- If $\vec{a}$ is any vector then unit vector is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

KEY POINTS

- If $\vec{a} = xi + yi + zk$ then its magnitude is given by

$$\left|\vec{a}\right| = \sqrt{x^2 + y^2 + z^2}$$

- If $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{b} = b_1 i + b_2 j + b_3 k$ are two vectors then

KEY POINTS

- The dot product is defined and denoted by

$$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- The cross product is defined and denoted by

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

SCALAR POINT FUNCTION

- If to each point of a region in space there corresponds a definite scalar is called **scalar point function**

Eg: The temperature at any instant

VECTOR POINT FUNCTION

- If to each point of a region in space there corresponds a definite vector is called **vector point function**

Eg: The velocity of a moving fluid at any instant

VECTOR DIFFERENTIAL OPERATOR

- The vector differential operator ∇ (read as del) is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

GRADIENT OF SCALAR POINT FUNCTION

- The gradient of a scalar point function is denoted by grad f or ∇f and is defined by

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

PROPERTIES OF GRADIENT

- If f and g are two scalar point functions then,
grad(f+g)=grad f + grad g
- If f is any constant scalar point function then
Grad f=0
- If f and g are two scalar point functions then,
grad(fg) = f grad g + g grad f

NORMAL AND UNIT NORMAL

- If f is any scalar point functions then, the normal to the surface f is given by

  grad f or ∇f

- The unit normal is given by

$$\frac{\nabla f}{|\nabla f|}$$

PROBLEMS

1. Find the normal to the surface $f = xy + yz$ at the point (1,1,1)

Sol: Given $f = xy + yz$

Normal to the surface f is given by

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = yi + (x+z)j + yk$$

Now,

$$(\nabla f)_{(1,1,1)} = i + 2j + k$$

PROBLEMS

2.Find the unit normal to the surface f at the point (1,1,1) given, $f = xy + yz$

Sol: Given $f = xy + yz$

Normal to the surface f is given by

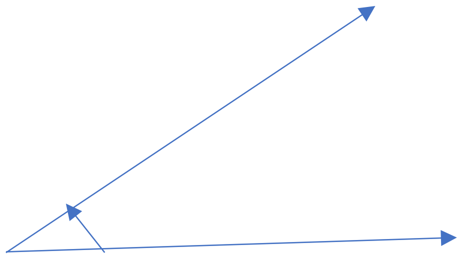
$$(\nabla f)_{(1,1,1)} = i + 2j + k$$

Unit normal =

$$\frac{\nabla f}{|\nabla f|} = \frac{i + 2j + k}{\sqrt{1+4+1}} = \frac{i + 2j + k}{\sqrt{6}}$$

1.The Angle between the vectors $\vec{A}$ and $\vec{B}$ is

$$Cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|}$$



i.e..

$$\theta = \cos^{-1}\frac{\vec{A}.\vec{B}}{|A||B|}$$

DOT PRODUCT:

- If $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{b} = b_1 i + b_2 j + b_3 k$ are two vectors then the dot product is defined and denoted by

$$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- The dot product produces a scalar quantity.
- It has no directions.
- $\vec{i}.\vec{i} = 1$
- $\vec{i}.\vec{j} = 0$
- $\vec{j}.\vec{j} = 1$
- $\vec{j}.\vec{k} = 0$
- $\vec{k}.\vec{k} = 1$
- $\vec{k}.\vec{i} = 0$
- The dot product is commutative v.w=w.v

- The Angle between the vectors A and B where $\vec{A} = 5\vec{i} - 3\vec{j} + 2\bar{k}$ and $\vec{B} = -2\vec{i} - 4\vec{j} + 3\vec{k}$

$$Cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|}$$

$$= \frac{(5\vec{i} - 3\vec{j} + 2\bar{k}).(-2\vec{i} - 4\vec{j} + 3\vec{k})}{\sqrt{5^2 + (-3)^2 + 2^2}.\sqrt{(-2)^2 + (-4)^2 + 3^2}}$$

$$= \frac{-10 + 12 + 6}{\sqrt{38}.\sqrt{29}} = \frac{8}{\sqrt{38}.\sqrt{29}}$$

$$\theta = \cos^{-1}\frac{8}{\sqrt{38}.\sqrt{29}}$$

5.The angle between the vectors A and B where $\vec{A} = \vec{i} + 2\vec{j} - \bar{k}$ and $\vec{B} = -4\vec{i} + \vec{j} - 2\vec{k}$

Sol: $Cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|}$

$$= \frac{(\vec{i}+2\vec{j}-\bar{k}).(-4\vec{i}+\vec{j}-2\vec{k})}{\sqrt{1^2+2^2+(-1)^2}.\sqrt{(-4)^2+(1)^2+(-2)^2}}$$

$$= \frac{-4+2+2}{\sqrt{6}\cdot\sqrt{21}} = 0$$

$$\cos\theta = 0$$

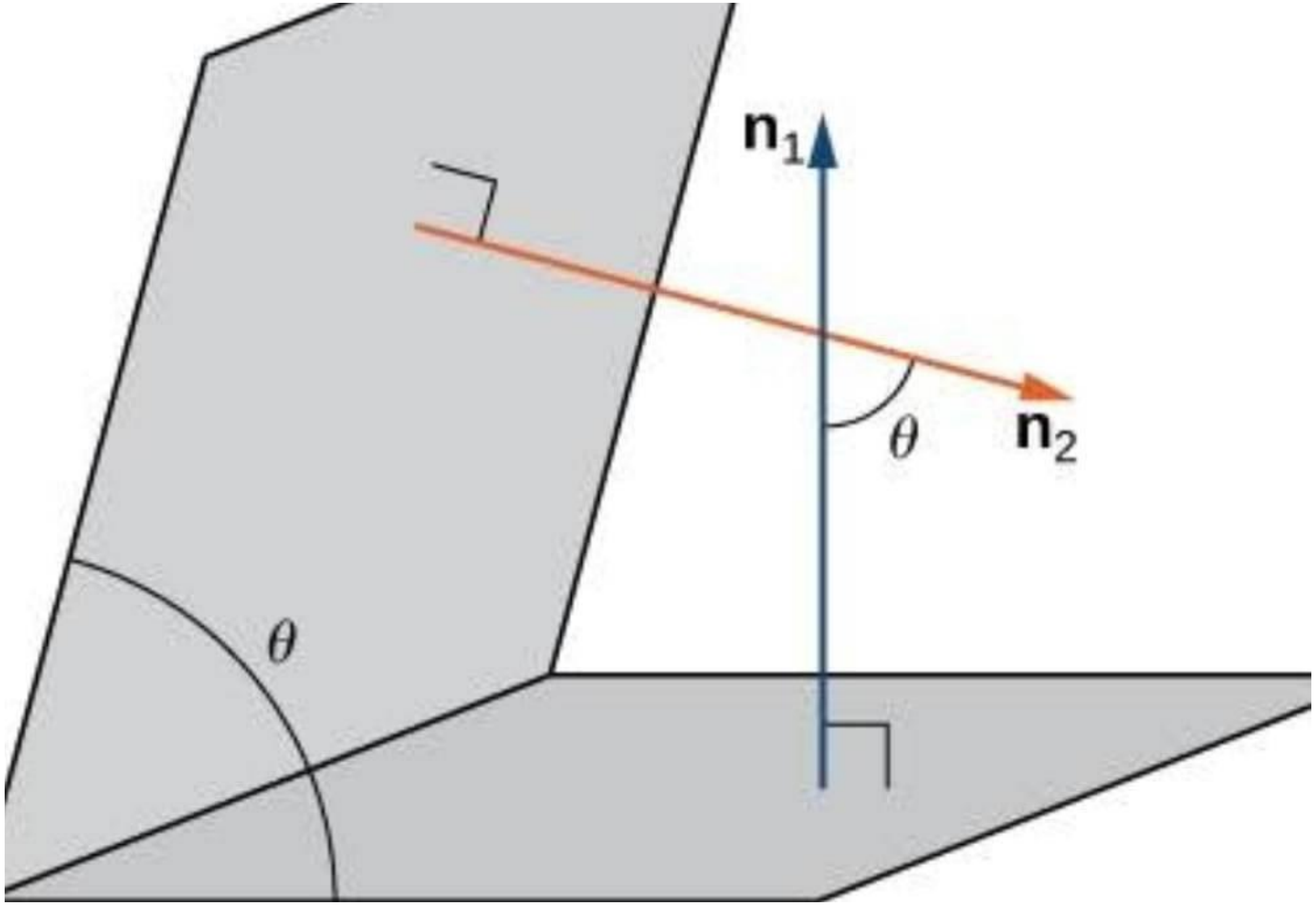
$$\theta = \cos^{-1} 0$$

$$\theta = 90^0$$

Angle between two surfaces

- Let f and g are two scalar point functions of the surfaces. If $\overline{n_1}$ *and* $\overline{n_2}$ are the normal vectors of that surfaces then the angle between these surfaces is

$$\cos\theta = \frac{\overline{n_1}\,.\,\overline{n_2}}{|\overline{n_1}|\,|\overline{n_2}|}$$



n1
n2
θ
θ

Problem:1

Find the angle between the normal to the surface $xy = z^2$ at the points (4,1,2) and (3, 3, −3).

Sol:

Given that $f = xy - z^2$

$$\text{grad f} = \bar{\imath}\frac{\partial f}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{\imath}\frac{\partial (xy - z^2)}{\partial x} + \bar{\jmath}\frac{\partial (xy - z^2)}{\partial y} + \bar{k}\frac{\partial (xy - z^2)}{\partial z}$$

$$= y\,\bar{\imath} + x\,\bar{\jmath} - 2z\bar{k}$$

- Normal vectors are

$$\overline{n_1} = (\,grad\ f)at\ (4,1,2) \quad = \ \bar{\imath} + 4\,\bar{\jmath}\ - 4\bar{k}$$

$$\overline{n_2} = (\,grad\ f)at\ (3,3,-3) = \ 3\bar{\imath} + 3\,\bar{\jmath} + 6\bar{k}$$

Let θ be the angle between the two normal vectors

$$\cos\theta = \frac{\overline{n_1}\ .\overline{n_2}}{|\overline{n_1}|\ |\overline{n_2}|}$$

$$\cos\theta = \frac{(\bar{\imath} + 4\,\bar{\jmath}\ - \ 4\bar{k}).\,(3\bar{\imath} + 3\,\bar{\jmath} + 6\bar{k})}{|\bar{\imath} + 4\,\bar{\jmath}\ - \ 4\bar{k}|\ |3\bar{\imath} + 3\,\bar{\jmath} + 6\bar{k}|}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\,\sqrt{54}} = \frac{-9}{\sqrt{33}\,\sqrt{54}}$$

Problem:2

Find the angle between the normal to the surfaces $x^2 + y^2 + z^2 = 9\ and\ z = x^2 + y^2 - 3$ at the point (2,-1,2).

Sol: Given that

$f = x^2 + y^2 + z^2 - 9$ and g=$x^2 + y^2 - z - 3$

Normal vectors are

$\overline{n_1}$ =grad f = $\bar{\imath}\,\frac{\partial f}{\partial x} + \bar{\jmath}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z}$

$$\overline{n_1} = \bar{\imath}\,\frac{\partial(x^2+y^2+z^2-9)}{\partial x} + \bar{\jmath}\,\frac{\partial(x^2+y^2+z^2-9)}{\partial y} + \bar{k}\,\frac{\partial(x^2+y^2+z^2-9)}{\partial z}$$

$$= 2x\,\bar{\imath} + 2y\,\bar{\jmath} + 2z\bar{k}$$

$\overline{n_1}$ at (2,-1,2) is $\overline{n_1}$ = $4\,\bar{\imath} - 2\,\bar{\jmath} + 4\,\bar{k}$

$$g = x^2 + y^2 - z - 3$$

$$\overline{n_2} = \text{grad } g = \bar{\imath}\,\frac{\partial g}{\partial x} + \bar{\jmath}\,\frac{\partial g}{\partial y} + \bar{k}\,\frac{\partial g}{\partial z}$$

$$\overline{n_2} = \bar{\imath}\,\frac{\partial(x^2+y^2-z-3)}{\partial x} + \bar{\jmath}\,\frac{\partial(x^2+y^2-z-3)}{\partial y} + \bar{k}\,\frac{\partial(x^2+y^2-z-3)}{\partial z}$$

$$= 2x\,\bar{\imath} + 2y\,\bar{\jmath} - \bar{k}$$

$\overline{n_2}$ at $(2,-1,2)$ is $\overline{n_2} = 4\,\bar{\imath} - 2\,\bar{\jmath} - \bar{k}$

$$\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}|\,|\overline{n_2}|}$$

$$\cos\theta = \frac{(4\,\bar{\imath} - 2\,\bar{\jmath} + 4\,\bar{k}).(4\,\bar{\imath} - 2\,\bar{\jmath} - \bar{k})}{|4\,\bar{\imath} - 2\,\bar{\jmath} + 4\,\bar{k}|\,|4\,\bar{\imath} - 2\,\bar{\jmath} - \bar{k}|}$$

$$\cos\theta = \frac{16+4-4}{\sqrt{36}\,\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Problem:3

Find the angle between the normal to the surfaces $x^2 + y^2 + z^2 = 29 \; and \; x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ at the point (4,-3,2).

Sol: Given that

$$f = x^2 + y^2 + z^2 - 29$$

Normal vectors are

$\overline{n_1}$ =grad f = $\bar{\imath}\,\frac{\partial f}{\partial x} + \bar{\jmath}\,\frac{\partial f}{\partial y} + \bar{k}\,\frac{\partial f}{\partial z}$

$\overline{n_1}$ = $\bar{\imath}\,\frac{\partial(x^2+y^2+z^2-29)}{\partial x} + \bar{\jmath}\,\frac{\partial(x^2+y^2+z^2-29)}{\partial y} + \bar{k}\,\frac{\partial(x^2+y^2+z^2-29)}{\partial z}$

= $2x\,\bar{\imath} + 2y\,\bar{\jmath} + 2z\bar{k}$

$\overline{n_1}$ at (4,-3,2) is $\overline{n_1}$ = $8\,\bar{\imath} - 6\,\bar{\jmath} + 4\,\bar{k}$

g = $x^2+y^2+z^2+4x-6y-8z-47=0$

$\overline{n_2}$ = grad g = $\bar{\imath}\,\frac{\partial g}{\partial x}+\bar{\jmath}\,\frac{\partial g}{\partial y}+\bar{k}\,\frac{\partial g}{\partial z}$

$\overline{n_2} = \bar{\imath}\,\frac{\partial(x^2+4x)}{\partial x}+\bar{\jmath}\,\frac{\partial(y^2-6y)}{\partial y}+\bar{k}\,\frac{\partial(z^2-8z)}{\partial z}$

$= (2x+4)\bar{\imath}+(2y-6)\bar{\jmath}+(2z-8)\bar{k}$

$\overline{n_2}$ at (4,-3,2) is $\overline{n_2}$ = $12\,\bar{\imath}-12\,\bar{\jmath}-4\bar{k}$

$$\cos\theta = \frac{\overline{n_1}\,.\overline{n_2}}{|\overline{n_1}|\,|\overline{n_2}|}$$

$$\cos\theta = \frac{(8\,\bar{\imath} - 6\,\bar{\jmath} + 4\,\bar{k}).(12\,\bar{\imath} - 12\,\bar{\jmath} - 4\bar{k})}{|8\,\bar{\imath} - 6\,\bar{\jmath} + 4\,\bar{k}|\,|12\,\bar{\imath} - 12\,\bar{\jmath} - 4\bar{k}|}$$

$$\cos\theta = \frac{96+72-16}{\sqrt{116}\,\sqrt{304}} = \frac{23}{\sqrt{551}}$$

$$\theta = \cos^{-1}\left(\frac{23}{\sqrt{551}}\right)$$

Practice Problem:

Find the angle between the normal to the surfaces $3x^2 - y^2 + 2z = 1 \; and \; xy^2z = z^2 + 3x$ at the point (1,-2,1).

Reference: Higher Engineering Mathematics, B.S. GREWAL , page no: 315 to 326.

Directional derivative

- The directional derivative of a scalar point function $\emptyset\ (x, y, z)$ at a point P(x, y, z) in the direction of a unit vector $\bar{e}$ is equal to

$$\bar{e}\,.\,grad\,\emptyset = \bar{e}\,.\,\nabla\emptyset$$

Note:

Let given surface is $\emptyset\ (x, y, z)$

The given direction vector $\bar{a}$

then the directional derivative is $\bar{e}\,.\,grad\,\emptyset$

where $\bar{e} = \dfrac{\bar{a}}{|\bar{a}|}$

1) Find the directional derivative of $f=x^2+y^2+z^2$ in the direction of vector $\bar{\imath}+2\bar{\jmath}+\bar{k}$ at the point (1,0,1)

Sol:

Given surface is $f=x^2+y^2+z^2$

$$\nabla f = \left(\bar{\imath}\frac{\partial}{\partial x}+\bar{\jmath}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)f$$

$$= \bar{\imath}\frac{\partial f}{\partial x}+\bar{\jmath}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{\imath}\frac{\partial(x^2+y^2+z^2)}{\partial x}+\bar{\jmath}\frac{\partial(x^2+y^2+z^2)}{\partial y}+\bar{k}\frac{\partial(x^2+y^2+z^2)}{\partial z}$$

$$= 2x\bar{\imath}+2y\bar{\jmath}+2z\bar{k}$$

en direction vector is $\bar{a} = \bar{i} + 2\bar{j} + \bar{k}$

unit normal vector is $\bar{e} = \frac{\bar{a}}{|\bar{a}|}$

$$= \frac{\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{1^2+2^2+1^2}}$$

$$= \frac{\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{6}}$$

Directional derivative of f along the given direction is $\bar{e}.\nabla f$

$$= \frac{\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{6}} \cdot 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$= \frac{2x+4y+2z}{\sqrt{6}} \text{ at } (1, 0, 1)$$

$$= \frac{2(1)+4(0)+2(1)}{\sqrt{6}} = \frac{4}{\sqrt{6}}$$

2) Find the directional derivative of

$f = xy + yz + xz$ in the direction of vector $\bar{\imath} + 2\bar{\jmath} + 2\bar{k}$ at the point (1,2,0)

Sol:

Given surface is $f = xy + yz + xz$

$$\nabla f = \left(\bar{\imath} \frac{\partial}{\partial x} + \bar{\jmath} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right)\emptyset$$

$$= \bar{\imath} \frac{\partial f}{\partial x} + \bar{\jmath} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z}$$

$$\nabla f = \bar{\imath} \frac{\partial(xy+yz+xz)}{\partial x} + \bar{\jmath} \frac{\partial(xy+yz+xz)}{\partial y} + \bar{k} \frac{\partial(xy+yz+xz)}{\partial z}$$

$$= (y+z)\bar{\imath} + (x+z)\bar{\jmath} + (y+x)\bar{k}$$

Given direction vector is $\bar{a} = \bar{\imath} + 2\bar{\jmath} + 2\bar{k}$

unit normal vector is $\bar{e} = \dfrac{\bar{a}}{|\bar{a}|}$

$$= \frac{\bar{\imath} + 2\bar{\jmath} + \mathbf{2}\bar{k}}{\sqrt{1^2+2^2+2^2}}$$

$$= \frac{\bar{\imath} + 2\bar{\jmath} + \mathbf{2}\bar{k}}{3}$$

Directional derivative of f along the given direction is $\bar{e}.\boldsymbol{\nabla} \boldsymbol{f}$

$$= \frac{\bar{\imath} + 2\bar{\jmath} + \mathbf{2}\bar{k}}{3} . (y+z)\bar{\imath} + (x+z)\bar{\jmath} + (y+x)\bar{k}$$

$$= \frac{(y+z)+2(x+z)+2(y+x)}{3} \text{ at } (1, 2, 0)$$

$$= \frac{(2+0)+2(1+0)+2(2+1)}{3} = \frac{10}{3}$$

3) Find the directional derivative of $f = 2xy + z^2$ in the direction of vector $\bar{\imath} + 2\bar{\jmath} + 3\bar{k}$ at the point (1,-1,3)

Sol:

Given surface is $f = 2xy + z^2$

$$\nabla f = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)f$$

$$= \bar{\imath}\frac{\partial f}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \bar{\imath}\frac{\partial(2xy+z^2)}{\partial x} + \bar{\jmath}\frac{\partial(2xy+z^2)}{\partial y} + \bar{k}\frac{\partial(2xy+z^2)}{\partial z}$$

$$= 2y\bar{\imath} + 2x\,\bar{\jmath} + 2z\,\bar{k}$$

Given direction vector is $\bar{a} = \bar{\imath} + 2\bar{\jmath} + 3\bar{k}$

unit normal vector is $\bar{e} = \dfrac{\bar{a}}{|\bar{a}|}$

$$= \frac{\bar{\imath} + 2\bar{\jmath} + \mathbf{3}\bar{k}}{\sqrt{1^2+2^2+3^2}}$$

$$= \frac{\bar{\imath} + 2\bar{\jmath} + \mathbf{3}\bar{k}}{\sqrt{14}}$$

Directional derivative of f along the given direction is $\bar{e} . \nabla f$

$$= \frac{\bar{\imath} + 2\bar{\jmath} + \mathbf{3}\bar{k}}{\sqrt{14}} . \left(2y\bar{\imath} + 2x\bar{\jmath} + 2z\bar{k}\right)$$

$$= \frac{2y+4x+6z}{\sqrt{14}} \text{ at } (1, -1, 3)$$

$$= \frac{2(-1)+4(1)+6(3)}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

4) Find the directional derivative of $f = x^2yz + 4xz^2$ in the direction of vector $2\bar{i} - \bar{j} - 2\bar{k}$ at the point (1,-2,-1)

Sol:

Given surface is $f = x^2yz + 4xz^2$

$$\nabla f = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)f$$

$$= \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}\frac{\partial(x^2yz+4xz^2)}{\partial x} + \bar{j}\frac{\partial(x^2yz+4xz^2)}{\partial y} + \bar{k}\frac{\partial(x^2yz+4xz^2)}{\partial z}$$

$= (2xyz+4z^2)\bar{i} + (x^2z)\bar{j} + (x^2y + 8xz)\bar{k}$

Given direction vector is $\bar{a} = 2\bar{\imath} - \bar{\jmath} - 2\bar{k}$

unit normal vector is $\bar{e} = \frac{\bar{a}}{|\bar{a}|}$

$$= \frac{2\bar{\imath} - \bar{\jmath} - 2\bar{k}}{\sqrt{2^2+(-1)^2+(-2)^2}}$$

$$= \frac{2\bar{\imath} - \bar{\jmath} - 2\bar{k}}{3}$$

Directional derivative of f along the given direction is $\bar{e}.\nabla f$

$$= \frac{2\bar{\imath} - \bar{\jmath} - 2\bar{k}}{3} \cdot (2xyz+4z^2)\bar{\imath} + (x^2z)\bar{\jmath} + (x^2y + 8xz)\bar{k}$$

$$= \frac{2(2xyz+4z^2) - (x^2z) - 2(x^2y+8xz)}{3} \text{ at } (1, -2, -1)$$

$$= \frac{2(4+4)-(-1)-2(-2-8)}{3} = \frac{37}{3}$$

Practice Problem:

Find the directional derivative of

$f = x^2yz^3$ in the direction of vector $2\bar{\imath} - \bar{\jmath} - 2\bar{k}$ at the point (2,1,-1)

Reference: Higher Engineering Mathematics, B.S. GREWAL , page no: 315 to 328.

NORMAL AND UNIT NORMAL

- If f is any scalar point functions then, the normal to the surface f is given by

  grad f or ∇f

- The unit normal is given by

$$\frac{\nabla f}{|\nabla f|}$$

PROBLEM:

Find the directional derivative of $\phi = xyz$ in the direction of the normal to the surface $x^2z + y^2x + yz^2 = 3$ at the point (1,1,1)

Solution: Given $\phi = xyz$

$$grad\phi = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$$

$$= yz\vec{i} + zx\vec{j} + xy\vec{k}$$

$$(grad\phi)_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$$

Now let $f = x^2 z + y^2 x + yz^2 - 3$ be the surface

Normal to the surface f is given by

$$grad f = \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}$$

$$= (2xz + y^2)\vec{i} + (2yx + z^2)\vec{j} + (x^2 + 2yz)\vec{k}$$

$$(grad f)_{(1,1,1)} = \vec{n} = 3\vec{i} + 3\vec{j} + 3\vec{k}$$

Now the unit normal of f is given by

$$\vec{e} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i} + 3\vec{j} + 3\vec{k}}{\sqrt{3^2 + 3^2 + 3^2}} = \frac{3\vec{i} + 3\vec{j} + 3\vec{k}}{\sqrt{27}}$$

- Directional derivative of ϕ along the normal is given by

$$\vec{e}.grad\phi = \frac{(3\vec{i} + 3\vec{j} + 3\vec{k})}{\sqrt{27}} \cdot (\vec{i} + \vec{j} + \vec{k})$$

$$= \frac{3+3+3}{\sqrt{27}}$$

$$= \frac{9}{\sqrt{27}}$$

$$= \sqrt{3}$$

PROBLEM:

Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $x\log z - y^2$ at the point (-1,2,1)

Solution: Given $\phi = x^2yz + 4xz^2$

$$grad\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

$$(grad\phi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Now let $f = x\log z - y^2$ be the surface

Normal to the surface f is given by

$$grad f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$= \log z\vec{i} + -2y\vec{j} + \frac{x}{z}\vec{k}$$

$$(grad f)_{(-1,2,1)} = \vec{n} = -4\vec{j} - \vec{k}$$

Now the unit normal of f is given by

$$\vec{e} = \frac{\vec{n}}{|\vec{n}|} = \frac{-4\vec{j} - \vec{k}}{\sqrt{(-4)^2 + (-1)^2}} = \frac{-4\vec{j} - \vec{k}}{\sqrt{17}}$$

Directional derivative of ϕ along the normal is given by

$$\vec{e}.grad\phi = \frac{(-4\vec{j} - \vec{k})}{\sqrt{17}} \cdot (8\vec{i} - \vec{j} - 10\vec{k})$$

$$= \frac{4+10}{\sqrt{17}}$$

$$= \frac{14}{\sqrt{17}}$$

PROBLEM:

Find the directional derivative of $\phi = xyz^2 + xz$ at the point (1,1,1) in the direction of the normal to the surface $3xy^2 + y = z$ at the point (0,1,1)

Solution: Given $\phi = xyz^2 + xz$

$$grad\phi = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$$

$$= (yz^2 + z)\vec{i} + xz^2\vec{j} + (2xyz + x)\vec{k}$$

$$(grad\phi)_{(1,1,1)} = 2\vec{i} + \vec{j} + 3\vec{k}$$

Now let $f = 3xy^2 + y - z$ be the surface

Normal to the surface f is given by

$$grad f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$= 3y^2\vec{i} + (6xy+1)\vec{j} + (-1)\vec{k}$$

$$(grad f)_{(0,1,1)} = \vec{n} = 3\vec{i} + \vec{j} - \vec{k}$$

Now the unit normal of f is given by

$$\vec{e} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i} + \vec{j} - \vec{k}}{\sqrt{(3)^2 + (1)^2 + (-1)^2}} = \frac{3\vec{i} + \vec{j} - \vec{k}}{\sqrt{11}}$$

- Directional derivative of ϕ along the normal is given by

$$\vec{e}.grad\phi = \frac{(3\vec{i} + \vec{j} - \vec{k})}{\sqrt{11}} \cdot (2\vec{i} + \vec{j} + 3\vec{k})$$

$$= \frac{6+1-3}{\sqrt{11}}$$

$$= \frac{4}{\sqrt{11}}$$

PROBLEM:

Find the directional derivative of $\phi = xyz$ in the direction of the normal to the surface $x^2 + y^2 + z^2 = 1$ at the point (0,1,1)

Solution: Given $\phi = xyz$

$$grad\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$(grad\phi)_{(0,1,1)} = \vec{i}$$

Now let $f = x^2 + y^2 + z^2 - 1$ be the surface

Normal to the surface f is given by

$$grad f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$(grad f)_{(0,1,1)} = \vec{n} = 2\vec{j} + 2\vec{k}$$

Now the unit normal of f is given by

$$\vec{e} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\vec{j} + 2\vec{k}}{\sqrt{(2)^2 + (2)^2}} = \frac{2\vec{j} + 2\vec{k}}{\sqrt{8}}$$

- Directional derivative of ϕ along the normal is given by

$$\vec{e}.grad\phi = \frac{(2\vec{j} + 2\vec{k})}{\sqrt{8}} \cdot (\vec{i})$$

$$= \frac{0 + 0 + 0}{\sqrt{8}}$$

$$= 0$$

Practice Problem:

- Find the directional derivative of $\phi = x^3 + yz^2$ at the point (0,1,1) in the direction of the normal to the surface $x + y + z = 2$ at the point (2,1,1).

1) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point (1,1,1)

Sol: Let $f = xy^2 + yz^2 + zx^2$

Directional Derivative=grad f.$\bar{e}$

grad f=Δf= $\vec{\imath}\,\frac{\partial f}{\partial x} + \bar{\jmath}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}$

$= \vec{\imath}\,\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + \bar{\jmath}\,\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2)$

$+ \vec{k}\,\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$

$= \vec{i}\,[y^2 + 2xz] + \vec{j}[2xy + z^2] + \vec{k}[2yz + x^2]$

$(\text{grad } f)_{(1,1,1)} = 3\,\vec{i} + 3\,\vec{j} + 3\,\vec{k}$

Let $\vec{r}$ be the position vector of any point on the curve $x = t, y = t^2, z = t^3$ then

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= t\vec{i} + t^2\vec{j} + t^3\,\vec{k}$$

$\frac{d\bar{r}}{dt}$ is the vector along with the tangent to the curve

$$\frac{d\bar{r}}{dt} = \vec{i} + 2t\vec{j} + 3t^2\,\vec{k} = \vec{i} + 2\vec{j} + 3\,\vec{k} \quad \text{at} \quad (1,1,1)$$

Unit vector along with the tangent

$$\bar{e} = \frac{\vec{i}+2\vec{j}+3\,\vec{k}}{\sqrt{1^2+2^2+3^2}} = \frac{\vec{i}+2\vec{j}+3\,\vec{k}}{\sqrt{14}}$$

Directional Derivative along with the tangent

=grad f.$\bar{e}$

$$= 3\,\vec{i} + 3\,\vec{j} + 3\,\vec{k}\,.\,\frac{\vec{i}+2\vec{j}+3\,\vec{k}}{\sqrt{14}}$$

$$= \frac{3+6+9}{\sqrt{14}} \qquad = \frac{18}{\sqrt{14}}\,..$$

2) Find the directional derivative of the function f=$x^2 - y^2 + 2z^2$ at the point P=(1,2,3) in the direction of the line $\overline{PQ}$ where Q=(5,0,4).

Sol:

Given f=$x^2 - y^2 + 2z^2$

grad f= Δf= $\vec{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}$

$= \vec{i}\,\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \bar{j}\,\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \vec{k}\,\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$

= $\vec{i}$ [2x] + $\bar{j}$ [-2y] + $\vec{k}$ [4z]

The position vectors of P and Q with respect to the origin are

$$\overline{OP} = \bar{\imath} + 2\bar{\jmath} + 3\bar{k}$$

$$\overline{OQ} = 5\bar{\imath} + 4\bar{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$

$$= (5\bar{\imath} + 4\bar{k}) - (\bar{\imath} + 2\bar{\jmath} + 3\bar{k})$$

$$= (4\bar{\imath} - 2\bar{\jmath} + \bar{k})$$

then $\bar{e} = \dfrac{(4\bar{\imath} - 2\bar{\jmath} + \bar{k})}{\sqrt{(4)^2+(-2)^2+(1)^2}} = \dfrac{(4\bar{\imath} - 2\bar{\jmath} + \bar{k})}{\sqrt{21}}$

The directional Derivative of $\bar{f}$ at P(1,2,3) in the direction of $\overline{PQ}$ is

$= \text{grad } f.\bar{e}$

$= (2x\,\vec{i} - 2y\,\bar{j} + 4z\,\vec{k}) \cdot \dfrac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}}$

$= \dfrac{8x+4y+4z}{\sqrt{21}}$ at (1,2,3)

$= \dfrac{28}{\sqrt{21}}$..

3) Find the directional derivative of $\emptyset = 5x^2y - 5y^2z + 2.5z^2x$ at the point P(1,1,1) in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = z$$

Sol)

Given $\emptyset = 5x^2y - 5y^2z + 2.5z^2x$

$$\nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(5x^2y - 5y^2z + 2.5z^2x) + \bar{j}\frac{\partial}{\partial y}(5x^2y - 5y^2z + 2.5z^2x) + \vec{k}\frac{\partial}{\partial z}(5x^2y - 5y^2z + 2.5z^2x)$$

$= \bar{i}[10xy + 2.5z^2] + \bar{j}[5x^2 - 10yz] + \bar{k}[-5y^2 + 5zx]$

$(\nabla\emptyset)_{(1,1,1)} = 12.5\,\bar{i} - 5\,\bar{j}$

Let (2,-2,1) be a point on the line then

$\bar{a} = (2\bar{i} - 2\bar{j} + \bar{k})$

$\bar{e} = \frac{(2\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{(2)^2 + (-2)^2 + 1^2}} = \frac{(2\bar{i} - 2\bar{j} + \bar{k})}{3}$

Directional Derivative=grad$\emptyset$. $\bar{e}$

$= (12.5\,\bar{i} - 5\,\bar{j}) \cdot \frac{(2\bar{i} - 2\bar{j} + \bar{k})}{3}$

$= \frac{25+10}{3} = \frac{35}{3}$..

4) Find the Directional Derivative of $\emptyset = x^4 + y^4 + z^4$ at the point A(1,-2,1) in the direction AB where B=(2,6,-1)

Sol)

Given $\emptyset = x^4 + y^4 + z^4$

$$grad\ \emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\ \frac{\partial\emptyset}{\partial z}$$

$$= \vec{i}[4x^3] + \bar{j}[4y^3] + \ \vec{k}[4z^3]$$

$$(grad\ \emptyset)_{(1,-2,1)} = 4\ \vec{i}\ - 32\bar{j} + 4\bar{k}$$

The position vectors of A and B with respect to the origin are

$$\overline{OA} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$$

$$\overline{OB} = 2\bar{\imath} + 6\bar{\jmath} - \bar{k}$$

$$\overline{AB} = \overline{OB} - \overline{OA}$$

$$= (2\bar{\imath} + 6\bar{\jmath} - \bar{k}) - (\bar{\imath} - 2\bar{\jmath} + \bar{k})$$

$$= \bar{\imath} + 8\bar{\jmath} - 2\bar{k}$$

$$\bar{e} = \frac{\bar{\imath}+8\bar{\jmath}-2\bar{k}}{\sqrt{(1)^2+(8)^2+(-2)^2}} = \frac{\bar{\imath}+8\bar{\jmath}-2\bar{k}}{\sqrt{69}}$$

Directional Derivation of ∅ in the direction of $\overline{AB}$ at (1,-2,1) is

$$grad\ \emptyset . \bar{e}$$

$$= (4\vec{i} - 32\bar{j} + 4\bar{k}) \ . \frac{\bar{i}+8\bar{j}-2\bar{k}}{\sqrt{69}}$$

$$= \frac{4-256-8}{\sqrt{69}}$$

$$= \frac{-260}{\sqrt{69}}\ldots$$

Practice Problem:

Find the Directional Derivative of $\emptyset$(x,y,z)= $4xy^2 + 2x^2yz$ at the point A(1,2,3) in the direction of the line AB where B=(5,0,4).

Divergence of a vector

- Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a vector point function then the divergence of a vector is denoted by $div\ \bar{f}$ and is defined as

$$div\ \bar{f} = \nabla \cdot \bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note : Divergence of a vector point function is a scalar point function

Properties:

- $div\,(\bar{f} \pm \bar{g}) = \mathrm{div}\bar{f} \pm div\,\bar{g}$
- $div\,k\bar{f}$ = k $div\ f$
- If $\boldsymbol{div\,\bar{f} = 0}$ then the vector point function is said to be **solenoidal** vector

1) Find $div\ \bar{f}$ at (1, -1, 1) where $\bar{f} = xy^2\bar{\imath} + 2x^2yz\bar{\jmath} - 3yz^2\bar{k}$

Sol: Given vector point function is

$$\bar{f} = xy^2\bar{\imath} + 2x^2yz\bar{\jmath} - 3yz^2\bar{k}$$

$$div\ \bar{f} = \nabla \cdot \bar{f}$$

$$= \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (xy^2\bar{\imath} + 2x^2yz\bar{\jmath} - 3yz^2\bar{k})$$

$$= \frac{\partial(xy^2)}{\partial x} + \frac{\partial(2x^2yz)}{\partial y} + \frac{\partial(-3yz^2)}{\partial z}$$

$$= y^2 + 2x^2z - 6yz$$

$(div\ \bar{f})$ at (1, -1, 1) = 1 + 2 +6 = 9

2) Find $div\ \bar{f}$ at (1, 2, - 3) where $\bar{f} = xy\bar{i} + 2yz\bar{j} - 3yz^2\bar{k}$

Sol: Given vector point function is

$$\bar{f} = xy\bar{i} + 2yz\bar{j} - 3yz^2\bar{k}$$

$$div\ \bar{f} = \nabla . \bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) . xy\bar{i} + 2yz\bar{j} - 3yz^2\bar{k}$$

$$= \frac{\partial(xy)}{\partial x} + \frac{\partial(2yz)}{\partial y} + \frac{\partial(-3yz^2)}{\partial z}$$

$$= y + 2z - 6yz$$

$(div\ \bar{f})$ at (1, 2, -3) = 2 + 2(-3) -6(2)(-3) = 2- 6 +36 = 32

3) Prove that the vector point function $\bar{f}$ is solenoidal vector at (1, 0, -1) , where $\bar{f} = (x^2 - yz)\bar{\imath} + (y^2 - zx)\,\bar{\jmath} + (z^2 - xy)\,\bar{k}$

Sol: Given vector point function is

$$\bar{f} = (x^2 - yz)\bar{\imath} + (y^2 - zx)\,\bar{\jmath} + (z^2 - xy)\,\bar{k}$$

$$div\ \bar{f} = \nabla \cdot \bar{f}$$

$$= \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (x^2 - yz)\bar{\imath} + (y^2 - zx)\,\bar{\jmath} + (z^2 - xy)\,\bar{k}$$

$$= \frac{\partial(x^2 - yz)}{\partial x} + \frac{\partial(y^2 - zx)}{\partial y} + \frac{\partial(z^2 - xy)}{\partial z}$$

$$= 2x + 2y + 2z$$

$(div\ \bar{f})$ at (1, 0, -1) = 2 + 0 -2 = 0

Hence the given vector is solenoidal vector.

4) Prove that the vector point function $\bar{f}$ is solenoidal vector , where $\bar{f} = 3y^4z^2\bar{\imath} + z^3x^2\,\bar{\jmath} - 3x^2y^2\,\bar{k}$

Sol: Given vector point function is

$$\bar{f} = 3y^4z^2\bar{\imath} + z^3x^2\,\bar{\jmath} - 3x^2y^2\,\bar{k}$$

$$div\ \bar{f} = \nabla \cdot \bar{f}$$

$$= \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot 3y^4z^2\bar{\imath} + z^3x^2\,\bar{\jmath} - 3x^2y^2\,\bar{k}$$

$$= \frac{\partial(3y^4z^2)}{\partial x} + \frac{\partial(z^3x^2)}{\partial y} + \frac{\partial(-3x^2y^2)}{\partial z}$$

$$= 0 + 0 + 0$$

$$(div\ \bar{f}) = 0$$

Hence the given vector is solenoidal vector.

5) If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal vector then find p

Sol: Given vector point function is

$$\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$$

$$div\ \bar{f} = \nabla \cdot \bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$$

$$= \frac{\partial(x+3y)}{\partial x} + \frac{\partial(y-2z)}{\partial y} + \frac{\partial(x+pz)}{\partial z}$$

=1 +1 +p=2+p

Given that $\bar{f}$ is solenoidal then $div\ \bar{f} = 0$

2+p = 0

$\therefore$ P = -2

6) Find $div\ \bar{f}$, where $\bar{f}$= grad $(x^3 + y^3 + z^3 - 3xyz)$

Sol: Let $\emptyset = x^3 + y^3 + z^3 - 3xyz$

Grad $\emptyset$= $\vec{i}\,\frac{\partial\emptyset}{\partial x} + \bar{j}\,\frac{\partial\emptyset}{\partial y} + \vec{k}\,\frac{\partial\emptyset}{\partial z}$

$\bar{f} = \vec{i}\,\frac{\partial(x^3+y^3+z^3-3xyz)}{\partial x} + \bar{j}\,\frac{\partial(x^3+y^3+z^3-3xyz)}{\partial y} + \vec{k}\,\frac{\partial(x^3+y^3+z^3-3xyz)}{\partial z}$

$\bar{f} = \vec{i}\,(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$

$div\ \bar{f} = \nabla . \bar{f}$

$= (\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}) . (3x^2 - 3yz)\bar{i} + (3y^2 - 3zx)\,\bar{j} + (3z^2 - 3xy)\,\bar{k}$

$= \frac{\partial(3x^2-3yz)}{\partial x} + \frac{\partial(3y^2-3zx)}{\partial y} + \frac{\partial(3z^2-3xy)}{\partial z}$

=3x +3y +3z

7) Find $div\ \bar{r}$, where $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

Sol: Given vector point function is

$$\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$$

$$div\ \bar{f} = \nabla\ .\ \bar{f}$$

$$= \left(\bar{\imath}\ \frac{\partial}{\partial x} + \bar{\jmath}\ \frac{\partial}{\partial y} + \bar{k}\ \frac{\partial}{\partial z}\right).\ (x\bar{\imath} + y\bar{\jmath} + z\bar{k})$$

$$= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z}$$

=1+1+1 =3

Practice Problem:

Find $div\ \bar{f}$, where $\bar{f} = (x+3y)\bar{\imath} + (y - 2z)\,\bar{\jmath} + (x - 2z)\,\bar{k}$

Curl of a vector

If $\vec{f}$ is any continuously differentiable vector point function then curl of a vector $\vec{f}$ is denoted by $curl\vec{f}$ or $\nabla \times \vec{f}$ and is defined as

$$curl\vec{f} = \nabla \times \vec{f} = \vec{i} \times \frac{\partial \vec{f}}{\partial x} + \vec{j} \times \frac{\partial \vec{f}}{\partial y} + \vec{k} \times \frac{\partial \vec{f}}{\partial z}$$

$$= \sum \vec{i} \times \frac{\partial \vec{f}}{\partial x}$$

i.e., if $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ then

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note:

1) If $\vec{f}$ is constant vector then $curl\vec{f} = \vec{O}$
2) $curl(\vec{a} \pm \vec{b}) = curl\vec{a} \pm curl\vec{b}$

PROBLEM:

Find $curl\vec{f}$ for $\vec{f} = 2xz^2\vec{i} - yz\vec{j} + 3xz^3\vec{k}$

Solution: Given,

$$\vec{f} = 2xz^2\vec{i} - yz\vec{j} + 3xz^3\vec{k}$$

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \vec{i}[0 + y] - \vec{j}[3z^3 - 4xz] + \vec{k}[0 - 0]$$

$$= y\vec{i} + \vec{j}(4xz - 3z^3)$$

PROBLEM:

Find $curl\vec{f}$ for $\vec{f} = z\vec{i} + x\vec{j} + y\vec{k}$

Solution: Given,

$$\vec{f} = z\vec{i} + x\vec{j} + y\vec{k}$$

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \vec{i}[1-0] - \vec{j}[0-1] + \vec{k}[1-0]$$

$$= \vec{i} + \vec{j} + \vec{k}$$

PROBLEM:

Find $div\,curl\vec{f}$ for $\vec{f} = xyz\vec{i} + zx\vec{j} + x\vec{k}$

Solution: Given, $\vec{f} = xyz\vec{i} + zx\vec{j} + x\vec{k}$

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & zx & x \end{vmatrix}$$

$$= \vec{i}\,[0 - x] - \vec{j}[1 - xy] + \vec{k}[z - xz]$$

$$= -x\vec{i} + \vec{j}[xy - 1] + \vec{k}[z - xz]$$

Now

$$div\,curl\vec{f} = \nabla \cdot \nabla \times \vec{f}$$

$$= (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot (-x\vec{i} + \vec{j}[xy - 1] + \vec{k}[z - xz])$$

$$= \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(xy - 1) + \frac{\partial}{\partial z}(z - xz)$$

$$= -1 + x + 1 - x = 0$$

PROBLEM:

Find $curl\vec{f}$ where $\vec{f} = grad(x^3 + y^3 + z^3 - 3xyz)$

Solution: Let

$$\phi = x^3 + y^3 + z^3 - 3xyz$$

$$\therefore \vec{f} = grad\phi$$

$$= \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= 3(x^2 - yz)\vec{i} + 3(y^2 - zx)\vec{j} + \vec{k}3(z^2 - xy)$$

Now

$$curl\vec{f} = curl(grad\phi) = \nabla \times grad\phi$$

$$= 3\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= 3[\vec{i}(-x + x) - \vec{j}(-y + y) + \vec{k}(-z + z)]$$

$$= \vec{0}$$

PROBLEM:

Find $\vec{f} \cdot curl\vec{f}$ for $\vec{f} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$

Solution: Given,

$$\vec{f} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$$

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \vec{i}\,[-1-0] - \vec{j}\,[-1-0)] + \vec{k}\,[0-1]$$

$$= -\vec{i} + \vec{j} - \vec{k}$$

Now

$$\vec{f} \cdot curl\vec{f} = \vec{f} \cdot \nabla \times \vec{f}$$

$$= [(x + y + 1)\vec{i} + \vec{j} - (x + y)\vec{k}] \cdot [-\vec{i} + \vec{j} - \vec{k}]$$

$$= -x - y - 1 + 1 + x + y = 0$$

Irrotational vector

If $\vec{f}$ is any continuously differentiable vector point function and $curl\vec{f} = \vec{0}$ then $\vec{f}$ is said to be irrotational

PROBLEM:

Show that the vector

$$\vec{f} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

is irrotational

Solution: Given

$$\vec{f} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \vec{i}\,[-x + x] - \vec{j}\,[-y + y] + \vec{k}\,[-z + z]$$

$$= \vec{0}$$

Hence given vector is irrotational.

PROBLEM:

Find the constants a,b,c if the vector

$$\vec{f} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+3z)\vec{k}$$

is irrotational

Solution: Given

$$\vec{f} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+3z)\vec{k}$$

is irrotational

$$curl\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\,[c-3] - \vec{j}\,[2-a] + \vec{k}\,[b-3] = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\Rightarrow (c-3)=0, -(2-a)=0, (b-3)=0$$

$$\Rightarrow c=3, a=-2, b=3$$

PRACTICE PROBLEMS:

1.Find $curl\vec{f}$ for $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

2. Find $curl\vec{r}$ where $\vec{r}$ is position vector

Scalar Potential:

A vector point function $\bar{f}$ is said to be irrotational if curl $\bar{f}=\bar{0}$.

If $\bar{f}$ is irrotational, then there exist a scalar function $\emptyset(x,y,z)$ such that $\bar{f}$=grad$\emptyset$. This '$\emptyset$' is called Scalar potential of $\bar{f}$.

Problem:1

Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find its Scalar potential.

Sol:

$$\text{Let } \bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$$

$$curl\ \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$curl\ \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$=\bar{i}\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right] - \bar{j}\left[\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right] + \bar{k}\left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]$$

$$= \bar{i}\ [-x + x] - \bar{j}[-y + y] + \bar{k}[-z + z]$$

$=\bar{0}$.

Curl $\bar{f}=\bar{0}$.

So, $\bar{f}$ is irrotational.

Then there exists $\emptyset$ such that $\bar{f}$=grad $\emptyset$.

$$(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k} = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

Comparing the components ,we get

$$\frac{\partial \emptyset}{\partial x} = x^2 - yz \;\; => \emptyset = \int (x^2 - yz)\; dx = \frac{x^3}{3} - xyz + f_1(y,z)$$

$$\frac{\partial \emptyset}{\partial y} = y^2 - zx \;\; => \emptyset = \int (y^2 - zx)\; dy = \frac{y^3}{3} - xyz + f_2(x,z)$$

$$\frac{\partial \emptyset}{\partial z} = z^2 - xy => \emptyset = \int (z^2 - xy)\; dz = \frac{z^3}{3} - xyz + f_3(x,y)$$

$$\emptyset = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$$

which is the required Scalar Potential.

2) Find the constants a, b, c if the vector

$$\vec{f} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$$

is irrotational. Also find $\emptyset$ such that $\bar{f} = \nabla\emptyset$

Sol: Given vector

$$\vec{f} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$$

$$curl\vec{f}\,(or)\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = \bar{0}$$

$$\Rightarrow \bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

Comparing both sides,

$c+1=0, \quad a-4=0, \quad b-2=0$

$c=-1, \quad a=4, \quad b=2$

Now, $\bar{A} = (x + 2y + az)\bar{i} + (bx - 3y - z)\bar{j} + (4x + cy + 2z)\bar{k}$ on substituting the values of a=4 ,b=2 ,c=-1

$$\bar{A} = (x + 2y + 4z)\bar{i} + (2x - 3y - z)\bar{j} + (4x - y + 2z)\bar{k}$$

Then there exists $\emptyset$ such that $\bar{A}$=grad $\emptyset$.

$$(x + 2y + 4z)\bar{i} + (2x - 3y - z)\bar{j} + (4x - y + 2z)\bar{k} = \bar{i}\,\frac{\partial\emptyset}{\partial x} + \bar{j}\,\frac{\partial\emptyset}{\partial y}\,\bar{k}\,\frac{\partial\emptyset}{\partial z}$$

$$(x+2y+4z)\bar{i}+(2x-3y-z)\bar{j}+(4x-y+2z)\bar{k}$$
$$=\bar{i}\,\frac{\partial\emptyset}{\partial x}+\bar{j}\,\frac{\partial\emptyset}{\partial y}\,\bar{k}\,\frac{\partial\emptyset}{\partial z}$$

Comparing on both sides, we have

$$\frac{\partial\emptyset}{\partial x}=x+2y+4z \Rightarrow \emptyset=\frac{x^2}{2}+2xy+4zx+f_1(y,z)$$

$$\frac{\partial\emptyset}{\partial y}=x-3y-z \Rightarrow \emptyset=2xy-\frac{3y^2}{2}-yz+f_2(z,x)$$

$$\frac{\partial\emptyset}{\partial z}=4x-y+2z \Rightarrow \emptyset=4xz-yz+z^2+f_3(x,y)$$

Hence $\emptyset = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - yz + c$

(or) $\emptyset = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx + c$

3) Prove that if $\bar{r}$ is the position vector of any point in space, then $r^n\bar{r}$ is irrotational

(or) Show that curl$(r^n\bar{r})= \bar{0}$.

Sol: Let $\bar{r}=x\,\bar{i} + y\bar{j} + z\bar{k}$ and r $= |\bar{r}|$

$$r^2 = x^2 + y^2 + z^2.$$

Differentiating w.r.t 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad => \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

We have $r^n\bar{r} = r^n\ (x\,\bar{i} + y\bar{j} + z\bar{k})$

$$\nabla \times (r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \bar{i}\{\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n\ y)\} - \bar{j}\{\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\} + \bar{k}\{\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\}$$

$$= \sum \bar{i}\ \{znr^{n-1}\frac{\partial r}{\partial y} - yn\ r^{n-1}\frac{\partial r}{\partial z}\}$$

$$= \mathrm{n}\, r^{n-1} \sum \bar{i} \left\{ z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right) \right\}$$

$$= \mathrm{n}\, r^{n-2} [(zy - yz)\, \bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}]$$

$$= \mathrm{n}\, r^{n-2} [0\bar{i} + 0\, \bar{j} + 0\bar{k}]$$

$$= \mathrm{n}\, r^{n-2} [\bar{0}] = \bar{0}$$

Hence $r^n \bar{r}$ is irrotational..

4) Show that F= $(2xy+z^3)\bar{i}+x^2\bar{j}+3xz^2\bar{k}$ is conservative force field and find the Scalar potential.

Sol: Let F= $(2xy+z^3)\bar{i}+x^2\bar{j}+3xz^2\bar{k}$

$$curl\ F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$curl\ F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right] - \bar{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right] + \bar{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= \bar{i}\ [0 - 0] - \bar{j}[3z^2 - 3z^2] + \bar{k}[2X - 2X]$$

$$= \bar{0}.$$

Curl F= $\bar{0}$.

So, F is irrotational, hence F is conservative

.

A vector F is conservative if their exists a scalar function $\emptyset$ such that F=$\nabla\emptyset$.

Let $\emptyset(x, y, z)$ be a scalar function then,

$$\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\bar{i} + \frac{\partial\emptyset}{\partial y}\bar{j} + \frac{\partial\emptyset}{\partial z}\bar{k}$$

$$F = \frac{\partial\emptyset}{\partial x}\bar{i} + \frac{\partial\emptyset}{\partial y}\bar{j} + \frac{\partial\emptyset}{\partial z}\bar{k}$$

$$(2xy+z^3)\bar{i}+x^2\bar{j}+3xz^2\bar{k}=\frac{\partial\phi}{\partial x}\bar{i}+\frac{\partial\phi}{\partial y}\bar{j}+\frac{\partial\phi}{\partial z}\bar{k}\ .$$

Comparing on both sides,

$$\frac{\partial\phi}{\partial x}=2xy+z^3\ ;\qquad \frac{\partial\phi}{\partial y}=x^2;\qquad \frac{\partial\phi}{\partial z}=3xz^2$$

$$d\emptyset=\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz$$

$$d\emptyset=(2xy+z^3)dx+x^2\,dy+3xz^2\,dz$$

$$d\emptyset=d(xz^3)+d(x^2y)$$

Integrating on both sides

$$\emptyset=xz^3+x^2y+c$$

5) Show that the vector field $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\bar{k}$ is irrotational. Find the scalar potential function.

Sol:

Given $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\bar{k}$

$$curl\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \bar{i}\,[2x^2z - y\sin(yz)(z) + \cos yz - 2x^2z + z\cos yz(y) - \cos yz]$$
$$+ \bar{j}\,[4xyz - 4xyz] + \bar{k}\,[2xz^2 - 2xz^2]$$

$$= \bar{0}.$$

$\therefore$ The function is irrotational.

There exists a scalar potential function $\emptyset$ such that grad $\emptyset = \bar{f}$.

$$\frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\bar{k}$$

Comparing the components,

$$\frac{\partial \phi}{\partial x} = 2xyz^2 \quad \Rightarrow \phi = x^2yz^2 + c_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = x^2z^2 + z\cos yz \Rightarrow \phi = x^2\,z^2\,y + \frac{z(\sin yz)}{z} + c_2(x,z)$$

$$\frac{\partial \phi}{\partial z} = 2x^2yz + y\cos yz \Rightarrow \phi = x^2yz^2 + \frac{y(\sin yz)}{y} + c_3(x,y)$$

$$\therefore \phi = x^2yz^2 + \sin yz + C$$

6)Show that Curl grad $\emptyset = 0$; where $\emptyset$ is a scalar function

Proof:

$$\text{grad}\,\emptyset = \frac{\partial\emptyset}{\partial x}\bar{i} + \frac{\partial\emptyset}{\partial y}\bar{j} + \frac{\partial\emptyset}{\partial z}\bar{k}.$$

$$\text{Curl}(\text{grad}\,\emptyset) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial x} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right) - \bar{j}\left(\frac{\partial^2\emptyset}{\partial x\partial z} - \frac{\partial^2\emptyset}{\partial z\partial x}\right) + \bar{k}\left(\frac{\partial^2\emptyset}{\partial x\partial y} - \frac{\partial^2\emptyset}{\partial y\partial x}\right)$$

$$= \bar{0}.$$

Note: grad $\emptyset$ is always irrotational.

PRACTICE PROBLEMS:

❖ Verify the vector $(x^2 - 3yz)\bar{i} + (y^2 - 3zx)\bar{j} + (z^2 - 3xy)\bar{k}$ is irrotational and also find its Scalar potential.

Vector operators

Vector differential operator ∇ :

The operator $\nabla = \bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$$\nabla\phi = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})\phi = \bar{\imath}\frac{\partial\phi}{\partial x} + \bar{\jmath}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla.\bar{f} = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\nabla \times \bar{f} = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}) \times \bar{f} = \bar{\imath} \times \frac{\partial \bar{f}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$$

Laplacian operator ∇^2:

The Laplacian operator is denoted by ∇^2 *and is defined as*

$$\nabla^2 \emptyset = \nabla . \nabla \emptyset$$

$$\nabla^2 \emptyset = \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) . \left(\bar{i} \frac{\partial \emptyset}{\partial x} + \bar{j} \frac{\partial \emptyset}{\partial y} + \bar{k} \frac{\partial \emptyset}{\partial z} \right)$$

$$\nabla^2 \emptyset = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2}$$

Note:

If $\nabla^2 \emptyset$ =0 then $\emptyset$ is said to satisfy Laplacian equation. This $\emptyset$ is called a harmonic function.

1) Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol:

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r $= |\bar{r}|$

$$r^2 = x^2 + y^2 + z^2.$$

Differentiating w.r.t 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now

$$\text{grad }(r^n) = \nabla (r^n) = \bar{i}\frac{\partial(r^n)}{\partial x} + \bar{j}\frac{\partial(r^n)}{\partial y} + \bar{k}\frac{\partial(r^n)}{\partial z}$$

$$= \bar{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \bar{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \bar{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= \bar{i}\, n r^{n-1}\left(\frac{x}{r}\right) + \bar{j}\, n r^{n-1}\left(\frac{y}{r}\right) + \bar{k}\, n r^{n-1}\left(\frac{z}{r}\right)$$

$$\text{grad }(r^n) = \bar{i}\, n x r^{n-2} + \bar{j}\, n y r^{n-2} + \bar{k}\, n z r^{n-2}$$

$\nabla^2 (r^n) = \nabla . (\nabla r^n) = \nabla . (\text{grad } r^n)$.

$= (\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}).(\bar{i} \, \mathrm{n}xr^{n-2} + \bar{j} \, \mathrm{n}yr^{n-2} + \bar{k} \, \mathrm{n}zr^{n-2})$

$= \frac{\partial(\mathrm{n}xr^{n-2})}{\partial x} + \frac{\partial(\mathrm{n}yr^{n-2})}{\partial x} + \frac{\partial(\mathrm{n}zr^{n-2})}{\partial x}$

$= \sum \frac{\partial(\mathrm{n}xr^{n-2})}{\partial x}$

$= \sum \{ \mathrm{n}r^{n-2} + nx\,(n-2)\, r^{n-3} \frac{\partial r}{\partial x} \}$

$$= \sum \{ nr^{n-2} + nx\,(n-2)\, r^{n-3} \left(\frac{x}{r}\right) \}$$

$$= \sum \{ nr^{n-2} + n\,(n-2) x^2 r^{n-4} \}$$

$$= nr^{n-2} + nr^{n-2} + nr^{n-2} + n\,(n-2)\, r^{n-4} (x^2 + y^2 + z^2)$$

$$= 3\, nr^{n-2} + n\,(n-2)\, r^{n-4}\, (r^2)$$

$$= 3\, nr^{n-2} + n\,(n-2)\, r^{n-2}$$

$$= nr^{n-2}\, (3 + n - 2) == n\,(n+1) r^{n-2}$$

2) Find $(\nabla \times A)\emptyset$, if $A = yz^2\bar{\imath} - 3xz^2\bar{\jmath} + 2xyz\bar{k}$ and $\emptyset = xyz$

Sol:

$$(\nabla \times A) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & -3xz^2 & 2xyz \end{vmatrix}$$

$$= \bar{\imath}\left\{\frac{\partial(2xyz)}{\partial y} - \frac{\partial(-3xz^2)}{\partial z}\right\} - \bar{\jmath}\left\{\frac{\partial(2xyz)}{\partial x} - \frac{\partial(yz^2)}{\partial z}\right\} + \bar{k}\left\{\frac{\partial(-3xz^2)}{\partial x} - \frac{\partial(yz^2)}{\partial y}\right\}$$

$$= \bar{\imath}\,(2xz+6xz) - \bar{\jmath}\,(2yz - 2yz) + \bar{k}\,(-3z^2 - z^2)$$

$$(\nabla \times A) = \bar{\imath}\,(8xz) - \bar{\jmath}\,(0) + \bar{k}\,(-4z^2)$$

$$(\nabla \times A) = \bar{i}\,(8xz) - \bar{j}\,(0) + \bar{k}\,(-4z^2)$$

$$(\nabla \times A)\emptyset = [\bar{i}\,(8xz) - \bar{j}\,(0) + \bar{k}\,(-4z^2)]\,xyz$$

$$= \bar{i}\,(8x^2yz) - \bar{j}\,(0) + \bar{k}\,(-4xyz^3)$$

3) Find $(A.\nabla)\emptyset$ at (1, -1, 1) if $A = 3xyz^2\bar{\imath} + 2xy^3\bar{\jmath} - x^2yz\bar{k}$ and $\emptyset = 3x^2 - yz$

Sol: Given that

$$A = 3xyz^2\bar{\imath} + 2xy^3\bar{\jmath} - x^2yz\bar{k}$$

$$(A.\nabla)\emptyset = \{ (3xyz^2\bar{\imath} + 2xy^3\bar{\jmath} - x^2yz\bar{k}).(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})\} 3x^2 - yz$$

$$= (3xyz^2\bar{\imath} + 2xy^3\bar{\jmath} - x^2yz\bar{k}).(\bar{\imath}\frac{\partial(3x^2-yz)}{\partial x} + \bar{\jmath}\frac{\partial(3x^2-yz)}{\partial y} + \bar{k}\frac{\partial(3x^2-yz)}{\partial z})$$

$(A.\nabla)\emptyset = (3xyz^2\bar{\imath} + 2xy^3\bar{\jmath} - x^2yz\bar{k}).\ (6x\,\bar{\imath} - z\bar{\jmath} - y\,\bar{k})$

$= 18x^2yz^2 - 2xy^3z + x^2y^2z$

$(A.\nabla)\emptyset$ at (1, -1, 1)=18(1)(-1)(1)-2(1)(-1)(1) + (1)(1)(1)

= - 18 + 2 +1

= -15

4) If $f = (x^2 + y^2 + z^2)^{-n}$ then find div(grad f) and determine n, if div(grad f) = 0

Sol: Given that $f = (x^2 + y^2 + z^2)^{-n}$

$$f(r) = (r^2)^{-n} = r^{-2n}$$

div(grad f)= $\nabla\,(\nabla\, f) = \nabla^2 f = \nabla^2\,(r^{-2n})$

But we know that (from problem 1)

$$\nabla^2(\,r^n) = n\,(n+1)r^{n-2}$$

Using this we write

$$\nabla^2\,(r^{-2n}) = -2n\,(-2n+1)r^{-2n-2}$$

div(grad f) = $\nabla^2\ (r^{-2n})$= $2n\ (2n-1)r^{-2n-2}$

Given that , div(grad f) = 0

$2n\ (2n-1)r^{-2n-2}$= 0

$\therefore n = 0\ (or)\ 2n - 1$=0

Hence n=0 (or) n=$\frac{1}{2}$

5) If $\emptyset$ satisfies Laplacian equation , prove that $\nabla\emptyset$ is both solenoidal and irrotational.

Sol:

Given that $\emptyset$ satisfies Laplacian equation

$$\therefore \nabla^2\emptyset = 0$$

$$\nabla.(\nabla\emptyset)=0$$

div (grad $\emptyset$)=0

Hence grad $\emptyset$ is solenoidal

Also we know that curl (grad $\emptyset$) =$\bar{0}$,where $\emptyset$ is any scalar function

Hence grad $\emptyset$ is always irrotational

Vector identities:

1) Prove that $div\ (\bar{a} \times \bar{b}) = \bar{b}.curl\bar{a} - \bar{a}.curl\bar{b}$

Proof:

$$div\ (\bar{a} \times \bar{b}) = \sum \bar{\imath}.\frac{\partial(\bar{a}\times\bar{b})}{\partial x}$$

$$= \sum \bar{\imath}.\left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum \bar{\imath}.\left(\frac{\partial \bar{a}}{\partial x} \times \bar{b}\right) + \sum \bar{\imath}.(\bar{a} \times \frac{\partial \bar{b}}{\partial x})$$

$$= \sum\left(\bar{\imath}\times\frac{\partial\bar{a}}{\partial x}\right).\bar{b} + \sum\left(\bar{\imath}\times\frac{\partial\bar{b}}{\partial x}\right).\bar{a}$$

$$= (\nabla\times\bar{a}).\bar{b} - (\nabla\times\bar{b}).\bar{a}$$

$$= \bar{b}.curl\bar{a} - \bar{a}.curl\,\bar{b}$$

Hence, $div\,(\bar{a}\times\bar{b}) = \bar{b}.curl\bar{a} - \bar{a}.curl\bar{b}$

2) Prove that curl $(\bar{a} \times \bar{b}) = \bar{a}.\,div\bar{b} - \bar{b}.\,div\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$

Proof: $curl(\bar{a} \times \bar{b}) = \sum \bar{\iota} \times \frac{\partial(\bar{a}\times\bar{b})}{\partial x}$

$$= \sum \bar{\iota} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum \bar{\iota} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b}\right) + \sum \bar{\iota} \times \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x}\right)$$

{ Since , $\boldsymbol{a \times (b \times c) = (a.c)b - (a.b)c}$ }

$$= \sum \left\{(\bar{\iota}.\bar{b})\frac{\partial \bar{a}}{\partial x} - \left(\bar{\iota}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b}\right\} + \sum \left\{\left(\bar{\iota}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - (\bar{\iota}.\bar{a})\frac{\partial \bar{b}}{\partial x}\right\}$$

$$= \sum\left\{(\bar{\imath}.\bar{b})\frac{\partial\bar{a}}{\partial x} - \left(\bar{\imath}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\} + \sum\left\{\left(\bar{\imath}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (\bar{\imath}.\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$= \sum(\bar{b}.\bar{\imath})\frac{\partial\bar{a}}{\partial x} - \sum\left(\bar{\imath}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{\imath}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - \sum(\bar{a}.\bar{\imath})\frac{\partial\bar{b}}{\partial x}$$

$$= (\bar{b}.\nabla)\bar{a} - (\nabla.\bar{a})\bar{b} + (\nabla.\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\bar{b}.\nabla)\bar{a} - (div\bar{a})\bar{b} + (div\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= (\bar{b}.\nabla)\bar{a} - (div\bar{a})\bar{b} + (div\bar{b})\bar{a} - (\bar{a}.\nabla)\bar{b}$$

$$= \bar{a}.div\bar{b} - \bar{b}.div\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

Hence,

$$\text{curl}\,(\bar{a} \times \bar{b}) = \bar{a}.div\bar{b} - \bar{b}.div\bar{a} + (\bar{b}.\nabla)\bar{a} - (\bar{a}.\nabla)\bar{b}$$

3) Prove that $(\nabla f \times \nabla g)$ is solenoidal

Sol: we know that $div\,(\bar{a} \times \bar{b}) = \bar{b}.\,curl\bar{a} - \bar{a}.\,curl\bar{b}$

Consider $\mathrm{div}\,(\nabla f \times \nabla g) = \nabla g.\,(\mathrm{curl}\ \nabla f)$ - ∇f.(curl ∇g)

= $\nabla g.\,[\mathrm{curl}(gradf)]$ - ∇f.[curl(grad g)]

= $\nabla g.$ [0]- ∇f.[0] = 0

Hence, $(\nabla f \times \nabla g)$ is solenoidal

PRATICE PROBLEM

1) Find $\nabla^2(r^3)$ where $r^2 = x^2 + y^2 + z^2$